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## Algebraic Curves and Balanced $n$ -ary Designs

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### Abstract

Let  $\mathcal{V}$  be a set of points and  $\mathcal{B}$  a collection of multi-subsets (called blocks) of  $\mathcal{V}$  each of size  $k$ . A balanced  $n$ -ary design is a pair  $(\mathcal{V}, \mathcal{B})$  such that each point occurs at most  $n - 1$  times in any block, and each unordered pair of distinct points occurs  $\lambda$  times in the blocks of  $\mathcal{B}$ . Note that in the block  $\{x, x, y\}$ , the pair  $\{x, y\}$  is counted twice in the block. We show here some constructions of balanced  $n$ -ary designs by using algebraic curves over finite fields.

## 1 Introduction

Let  $\mathcal{V}$  be a set of  $v$  points and  $\mathcal{B}$  a collection of multi-subsets, called blocks, of  $\mathcal{V}$ . A balanced  $n$ -ary design is a pair  $(\mathcal{V}, \mathcal{B})$  satisfying

- (1) each block is of a constant size  $k$ ,
- (2) each point occurs at most  $n - 1$  times in any block  $B \in \mathcal{B}$ , and
- (3) each unordered pair of distinct points occurs exactly  $\lambda$  times in the blocks of  $\mathcal{B}$ .

Note that, for example, the block size of  $B = \{x, x, x, y, y, z\}$  is 6 since the points  $x$ ,  $y$  and  $z$  occur 3 times, twice and once, respectively. And in the

block  $B$  the pairs  $\{x, y\}$ ,  $\{y, z\}$  and  $\{x, z\}$  are counted 6, 2 and 3 times, respectively.

Let  $N = (n_{ij})$  be a  $|\mathcal{V}| \times |\mathcal{B}|$  matrix, where  $n_{ij}$  is the number of occurrences of the  $i$ -th point in the  $j$ -th block. We consider  $N$  as the incidence matrix of a balanced  $n$ -ary design. Using the incidence matrix, the conditions in the definition of a balanced  $n$ -ary design can be rewritten as follows:

- (1')  $\sum_i n_{ij} = k$  for any  $j$ ,
- (2')  $0 \leq n_{ij} \leq n - 1$  for any  $i, j$ , and
- (3')  $\sum_j n_{ij} n_{i'j} = \lambda$  for any unordered pair  $\{i, i'\}$ ,  $i \neq i'$ .

**Example 1.1.** Let  $\mathcal{V} = \{a, b, c, d, e\}$  and  $\mathcal{B}$  be the collection of the following blocks:

$$\begin{aligned} &\{a, b, b, d, d, e, e, e\}, \{a, a, c, c, c, d, e, e\}, \\ &\{b, b, b, c, c, d, d, e\}, \{a, a, b, c, c, d, d, d\}, \\ &\{a, a, a, b, b, c, e, e\}, \{a, c, c, d, d, d, e, e\}, \\ &\{a, a, b, b, b, c, d, d\}, \{b, b, c, c, d, e, e, e\}, \\ &\{a, a, b, b, c, c, c, e\}, \{a, a, a, b, d, d, e, e\}. \end{aligned}$$

Then  $(\mathcal{V}, \mathcal{B})$  is a balanced 4-ary (quaternary) design with 5 points, 10 blocks

and the block size is 8. The incidence matrix of the above balanced 4-ary design is

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 1 & 2 & 0 & 2 & 3 \\ 2 & 0 & 3 & 1 & 2 & 0 & 3 & 2 & 2 & 1 \\ 0 & 3 & 2 & 2 & 1 & 2 & 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 0 & 2 & 2 & 0 & 3 & 1 & 2 \end{bmatrix},$$

and for any pair  $\{i, i'\}$ ,  $i \neq i'$ ,

$$\sum_j n_{ij}n_{i'j} = 23 = \lambda.$$

Let  $\rho_s^{(i)}$  be the number of blocks containing the  $i$ -th point exactly  $s$  times, i.e. the number of the entry  $s$  in the  $i$ -th row of the incidence matrix. If  $\rho_s^{(i)} = \rho_s$  for all  $i$  then the design is said to be *regular*. If a balanced  $n$ -ary design is regular then the replication number  $R_i$  of the  $i$ -th point is a constant number  $R$ , since

$$R_i = \sum_{r=0}^{n-1} r \rho_r^{(i)} = \sum_r r \rho_r = R.$$

Balanced  $n$ -ary designs were first introduced by Tocher [3] in a statistical paper. The interested reader is referred to [1, 2] for their excellent surveys.

## 2 Algebraic curves

Let  $q$  be a prime power, and  $GF(q)$  a finite field of order  $q$ . A *divisor*  $D$  on a curve  $C$  is a formal sum  $\sum_{P \in C} m_P P$ ,  $m_P \in \mathbb{Z}$ . The set of divisors on  $C$  is

denoted by  $\text{Div}(D)$ . The *support* of a divisor  $D$ , denoted by  $\text{Supp}(D)$ , is the set of points satisfying  $m_P \neq 0$ . A divisor  $D$  is said to be *efficient* if  $m_P \geq 0$  for all  $P$ , and denoted by  $D \geq 0$ . The *degree* of  $D$  is  $\deg D = \sum_P m_P$ . Let  $D = \sum_P m_P P$  and  $E = \sum_P m'_P P$ . Then  $D + E = \sum_P (m_P + m'_P) P$ .

Let  $\text{Rat}(C)$  be the set of rational functions over a curve  $C$ . The divisor of a rational function  $f \in \text{Rat}(C)$  is  $\text{div}(f) = \sum_{P \in C} m_P P$ , where  $m_P$  is the order of  $f$  at  $P$ . For a divisor  $D$ , a divisor  $E$  is said to be *equivalent* to  $D$ , denoted by  $E \sim D$ , if there exists a rational function  $f \in \text{Rat}(C)$  satisfying  $E = D + \text{div}(f)$ .

Let  $C$  be a curve defined by  $F = 0$ , where  $F$  is a polynomial over a finite field  $GF(q)$ . The divisor of the curve  $C$  is the divisor of  $F$ . Note that  $\text{div}(C) = \text{div}(F) \geq 0$ . When the divisor of  $C$  is written as  $\text{div}(C) = \sum_p m_p p$ , the *intersection multiplicity* of a point  $p$  on a curve  $C'$  with the curve  $C$  is the order of  $F$  at  $p$ , say  $m_p$ .

Let  $L(D) = \{f \in \text{Rat}(C) : \text{div}(f) + D \geq 0\} \cup \{0\}$ . It is well-known that  $L(D)$  is a linear space with a finite dimension over an extension field  $GF(q^m)$  of  $GF(q)$ .

### 3 Algebraic curves and balanced $n$ -ary designs

Let  $C$  be an irreducible curve defined over  $GF(q)$  and  $\mathcal{C} = \{C_1, \dots, C_b\}$  a set of  $b$  curves defined over an extension  $GF(q^m)$  of  $GF(q)$ . Let  $V_j$  be a set of intersection points of  $C_j \in \mathcal{C}$  with  $C$ , and  $\mathcal{V} = \bigcup_j V_j = \{p_1, \dots, p_v\}$ .

Let  $n_{ij}$  denotes the intersection multiplicity of a point  $p_i$  with a curve  $C_j$ . We consider the  $v \times b$  matrix  $(n_{ij})$  as the incidence matrix. To satisfy the conditions of balanced  $n$ -ary designs, we have to choose suitable  $C$ ,  $\mathcal{C}$  and  $\mathcal{V}$ .

The first condition is required for the block size to be constant, i.e.,  $\sum_i n_{ij} = k$  for any curve  $C_j \in \mathcal{C}$ . Let  $D$  be a divisor on  $C$  and  $\mathcal{D} = \{\text{div}(f) + D : f \in L(D) \setminus \{0\}\} = \{E_1, \dots, E_b\}$ . Note that each element of  $\mathcal{D}$  is the divisor of a curve.

**Lemma 3.1.** *Let  $\mathcal{C}$  be the set of curves such that their divisors are the elements of  $\mathcal{D}$ . For any curve of  $\mathcal{C}$ , the total number of multiplicities of its intersection points with  $C$  is a constant number  $k$ .*

*Proof.* The set  $\mathcal{D}$  is the set of efficient divisors being equivalent to  $D$ , i.e.,

$$\begin{aligned} \mathcal{D} &= \{\text{div}(f) + D : f \in L(D) \setminus \{0\}\} \\ &= \{E \in \text{Div}(C) : E \geq 0, E \sim D\}. \end{aligned}$$

It is well-known that if  $E \sim D$  then  $\deg D = \deg E$ . Hence for any  $j$

$$\sum_i n_{ij} = \deg E_j = \deg D = k.$$

□

We assume here that we choose  $\mathcal{C}$  as above, and that the point set  $\mathcal{V}$  of a design is the set  $\mathcal{V} = \bigcup_{E \in \mathcal{D}} \text{Supp}(E)$ . The second condition says that each point of the design occurs at most  $n - 1$  times. Since the intersection multiplicities of points on curves are always positive, this condition is automatically satisfied from Lemma 3.1.

**Theorem 3.2.** *Let  $C$  be a curve defined over a finite field,  $D$  a divisor on  $C$ ,  $\mathcal{D} = \{\text{div}(f) + D : f \in L(D) \setminus \{0\}\} = \{E_1, \dots, E_b\}$ ,  $\mathcal{V} = \bigcup_{E \in \mathcal{D}} \text{Supp}(E)$   $E_j = \sum n_{ij} P_i$ ,  $P_i \in \mathcal{V}$ . If  $\deg C \leq 2$  and  $D \geq 0$  then  $(n_{ij})$  is the incidence matrix of a balanced  $n$ -ary design  $(\mathcal{V}, \mathcal{D})$ .*

*Proof.* We only have to check whether the third condition of balanced  $n$ -ary designs is satisfied. Let  $E_j$  be the  $j$ -th element of  $\mathcal{D}$ . Assume that  $D$  and  $E_j$ 's have the forms  $D = \sum_i m_i P_i$  and  $E_j = \text{div}(f_j) + D$ , respectively. Let  $\text{div}(f_j) = \sum_i e_{ij} P_i$ . Then each entry  $n_{ij}$  of the incidence matrix  $(n_{ij})$  is  $m_i + e_{ij}$ , since  $\text{div}(f_j) + D = \sum_i e_{ij} P_i + D = \sum_i (m_i + e_{ij}) P_i$ . For any pair

$\{i, i'\}$ , we have

$$\begin{aligned}
& \sum_j n_{ij} n_{i'j} \\
&= \sum_j (m_i + e_{ij})(m_{i'} + e_{i'j}) \\
&= \sum_j (m_i m_{i'} + m_i e_{i'j} + m_{i'} e_{ij} + e_{ij} e_{i'j}) \\
&= b m_i m_{i'} + m_i \sum_j e_{i'j} + m_{i'} \sum_j e_{ij} + \sum_j e_{ij} e_{i'j}.
\end{aligned}$$

Let  $\lambda(a, b)$  be the number of  $j$  satisfying  $(e_{ij}, e_{i'j}) = (a, b)$ . Then we have

$$\sum_j e_{ij} e_{i'j} = \sum_{(a,b)} ab \lambda(a, b).$$

When the degree of base curve  $C$  is less than or equal to 2, it can be easily seen that if  $a + b = c + d$  then  $\lambda(a, b) = \lambda(c, d)$ . Moreover we can see that

$$\sum_j e_{i'j} = \sum_j e_{ij} = \sum_a a \lambda(a),$$

where  $\lambda(a)$  is the number of  $j$  satisfying  $n_{ij} = a$ . Since both of  $\lambda(a, b)$  and  $\lambda(a)$  are independent of  $\{i, i'\}$  chosen, we have

$$\sum_j n_{ij} n_{i'j} = \lambda,$$

and we can conclude that the third condition in the definition of a balanced  $n$ -ary design is also satisfied.  $\square$



## References

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